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Mean Boundary Value Problems and Riemann Series

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1. INTRODUCTION

Let f be a continuous function on the unit circle T: $|z| = 1$. We consider the arithmetic means of f

$$s_n(f) = 1/n \sum_{k=1}^{n} f(e^{i2\pi k/n}), \qquad n = 1, 2,...,$$

and

$$s_\infty(f) = \lim_{n\to\infty} s_n(f).$$

If f belongs to A, the space of all continuous functions g on T with the Fourier coefficients $a_n(g) = 0$ for all $n < 0$, it is trivial that the holomorphic extension F of f in the open unit disc U is determined by the values of f on a dense subset of T. In this paper, we obtain the function F from the means $s_n(f)$ of f on T:

$$F(z) = \sum_{n=1}^{\infty} \{s_n(f) - s_\infty(f)\} p_n(z) + s_\infty(f), \tag{1}$$

p_n being some polynomial of degree n, assuming that the function f is "smooth" (e.g., in $C^{1+\epsilon}(T)$). The coefficients $r_n(f) = s_n(f) - s_\infty(f)$, $n \geqslant 1$, and $r_0(f) = s_\infty(f)$ are called the Riemann coefficients of f (cf. [2]). The asymptotic similarities of $r_n(f)$ and the Fourier coefficients of f have been pointed out in [3] and studied in [2]. The behavior of the series (1) is quite peculiar; for instance, there exists a sequence $\{r_n\}$ such that $r_n = O(1/n)$ and the series $\sum r_n p_n(z)$ diverges everywhere inside the unit circle except at the origin. Since the averages $s_n(g)$ of a function g on T do not give any information about its odd part, we have to consider both $s_n(g)$ and $s_n(\partial g/\partial\theta)$ in order to recapture g. Then we establish the existence and uniqueness of a harmonic function u with prescribed means $s_n(u)$ and $s_n(u_\theta)$ on T. Also, we obtain some analogous results for the wave equations and heat equations with prescribed mean initial values. Neumann problems are also considered. Our representation theorem is also extended to the unit polydisc.

2. Representation of Holomorphic Functions

For each $\epsilon > 0$, let B_ϵ be the class of all continuous functions f on T such that the Fourier coefficients $a_n(f)$ of f on T satisfy $a_n(f) = O(1/n^{1+\epsilon})$. The following theorem is obtained in [1].

Theorem A. *If $f \in A \cap B_\epsilon$ for some $\epsilon > 0$ and $s_n(f) = 0$ for $n = 1, 2,\ldots,$ then f is the zero function.*

It is easy to show that for each n there exists a unique polynomial p_n of degree n, leading coefficient equal to one, and $p_n(0) = 0$, such that $r_m(p_n) = \delta_{m,n}$, the Kronecker delta, $m, n = 1, 2,\ldots$ (cf. [1]). We also let $p_0 = 1$. Hence, $r_m(p_n) = \delta_{m,n}$ for $m, n = 0, 1, 2,\ldots$.

Theorem 1. *Let $f \in A \cap B_\epsilon$ for some $\epsilon > 0$. Then the series $\sum_{n=0}^{\infty} r_n(f)\, p_n(z)$ converges uniformly to the holomorphic extension F of f in U. Furthermore, the following inequalities hold:*

$$\left| F(z) - \sum_{k=0}^{n} r_k(f)\, p_k(z) \right| \leqslant M(\delta, f)/n^\delta \tag{2}$$

for all $n \geqslant 1$, $\delta < \epsilon$ and $| z | \leqslant 1$, and

$$\left| F(z) - \sum_{k=0}^{n} r_k(f)\, p_k(z) \right| \leqslant \frac{K(f)\, | z |}{1 - | z |} \cdot \frac{1}{n^\epsilon} \tag{3}$$

for all $n \geqslant 1$ and $| z | < 1$.

We first prove the following lemma.

Lemma 1. *For $n > 0$, $p_n(z) = \sum_{k \mid n} \mu(n/k)\, z^k$. Here, as usual, $k \mid n$ means that k is a factor of n and $\mu(n)$ is the Möbius function of n*

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1, \\ (-1)^k & \text{if } n = q_1, \ldots, q_k, \\ 0 & \text{if } p^2 \mid n \text{ for some } p > 1, \end{cases}$$

where $q_1, \ldots, q_k$ are distinct primes.

To prove the lemma, we observe that for $n \geqslant 1$,

$$r_k(z^n) = \begin{cases} 1 & \text{if } k \mid n, \\ 0 & \text{otherwise,} \end{cases}$$

and from the definition of $p_n(z)$, we have

$$r_k\left[\sum_{j|n} p_j(z)\right] = \begin{cases} 1 & \text{if } k \mid n, \\ 0 & \text{otherwise.} \end{cases}$$

Hence, in view of Theorem A, we can deduce that

$$z^n = \sum_{j|n} p_j(z)$$

for $n = 1, 2, \ldots$. Now, apply the Möbius inversion theorem (cf. [4, p. 236]) to give

$$p_n(z) = \sum_{s|n} \mu(n/s)\, z^s,$$

$n = 1, 2, \ldots$ as asserted.

We can now prove Theorem 1. Let $d(n)$ denote the number of distinct divisors of n. It is well known (cf. [4]) that for each $\delta > 0$, $d(n) \leqslant C_\delta n^\delta$ for some constant C_δ and all n. Hence, from the above lemma, we have

$$|p_n(z)| \leqslant \sum_{s|n} |\mu(n/s)| \leqslant d(n) \leqslant C_\delta n^\delta \tag{4}$$

for $|z| \leqslant 1$ and $\delta > 0$. Since $f \in B_\epsilon$, $\epsilon > 0$, we can (cf. [2]) find a constant K such that

$$|r_n(f)| \leqslant K/n^{1+\epsilon} \tag{5}$$

for all n. Thus, by picking $0 < \delta < \epsilon$, we can conclude that the series $\sum r_n(f)\, p_n(z)$ converges uniformly on $\bar{U}$ to some function F, holomorphic in U and continuous on $\bar{U}$. As usual, let F^* be the restriction of F on T. Now, the Fourier coefficients of F^* are

$$\begin{aligned} a_m(F^*) &= a_m\left(\sum_{n=0}^{\infty} r_n(f) \sum_{j|n} \mu(n/j)\, z^j\right) \\ &= \sum_{k=1}^{\infty} r_{km}(f)\, \mu(k), \end{aligned} \tag{6}$$

so that

$$|a_m(F^*)| \leqslant K \sum_{k=1}^{\infty} \frac{1}{(km)^{1+\epsilon}} = \frac{C}{m^{1+\epsilon}}.$$

Hence, both f and F^* belong to $A \cap B_\epsilon$. Furthermore, for each $n = 0, 1, \ldots$

$$\begin{aligned} r_n(F^*) &= r_n\left(\sum_{k=0}^{\infty} r_k(f)\, p_k(z)\right) \\ &= \sum_{k=0}^{\infty} r_k(f)\, \delta_{n,k} = r_n(f), \end{aligned}$$

which implies $s_n(F^* - f) = 0$ for all $n = 1, 2, \ldots$. By Theorem A, $F^* = f$. To prove (2) and (3), we use (4) and (5) to obtain

$$\left| F(z) - \sum_{k=0}^{n} r_k(f)\, p_k(z) \right| = \left| \sum_{k=n+1}^{\infty} r_k(f)\, p_k(z) \right|$$

$$\leqslant \sum_{k=n+1}^{\infty} (K/n^{1+\epsilon})\, C_{\epsilon-\delta} n^{\epsilon-\delta} = C/n^{\delta}$$

for all $|z| \leqslant 1$, and

$$\left| F(z) - \sum_{k=0}^{n} r_k(f)\, p_k(z) \right| \leqslant \sum_{k=n+1}^{\infty} |r_k(f)| \sum_{j|k} |z|^j$$

$$\leqslant \frac{C|z|}{1-|z|} \cdot \frac{1}{n^{\epsilon}}$$

for all $|z| < 1$. This completes the proof of Theorem 1.

As mentioned in the introduction, for a "smooth" function f on T, the Riemann coefficients $r_n(f)$ and the Fourier coefficients $a_n(f)$ behave very much alike as n tends to infinity. Hence, we would like to study the series

$$\sum_{n=0}^{\infty} r_n p_n(z), \tag{7}$$

where $\{r_n\}$ is a sequence of complex numbers. We shall call (7) a *Riemann series*.

THEOREM 2. (*i*) *If* $\{r_n\}$ *is a sequence of complex numbers such that* $\sum |r_n| d_n < \infty$, *then the series* (7) *converges uniformly on* $\overline{U}$ *to a function* F *in* A *with Riemann coefficients* $r_n(F^*) = r_n$ *for all* n.

(i′) *If* $r_n = O(1/n^{1+\epsilon})$ *for some* $\epsilon > 0$, *then the above conclusion also holds.*

(ii) *There exists a sequence* $r_n = O(1/n)$ *such that the series* (7) *diverges everywhere in* U *except at the origin.*

(iii) *For an integer* $q > 1$ *and* $\sum |r_n| < \infty$, *the series* $\sum r_k p_{q^k}(z)$ *converges uniformly on* $\overline{U}$ *to a function* F *in* A *with Riemann coefficients* $r_n(F^*) = r_k$ *if* $n = q^k$ *and* $r_n(F^*) = 0$ *otherwise.*

(iv) *Let* $\{r_n\}$ *be a sequence of real numbers, monotonically decreasing to zero, such that* $\sum r_n \log n < \infty$. *Then the series* (7) *converges uniformly on* $\overline{U}$.

Parts (i), and hence (i′), follow easily from the proof of Theorem 1. For the proof of (ii), we let

$$r_k = \begin{cases} 1/k & \text{if } k \text{ is a prime,} \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\sum_{k=1}^{\infty} r_k p_k(z) = \sum_{k \text{ prime}} (1/k)(z^k - z)$$

which is divergent for $0 < |z| < 1$. To prove (iii), we observe that

$$\begin{aligned} |p_{q^k}(z)| &= \left| \sum_{j|q^k} \mu(j)\, z^{q^k/j} \right| \\ &= \left| \sum_{j|q} \mu(j)\, z^{q^k/j} \right| \\ &\leqslant \sum_{j|q} |\mu(j)| \leqslant d(q) \end{aligned}$$

for all $|z| \leqslant 1$. Hence, since $\sum |r_k| < \infty$, $\sum r_k p_{q_k}(z)$ converges uniformly on $\bar{U}$ to a function F in A as asserted. Here, the function F is uniquely determined by the sequence $\{r_k\}$, (cf. [1]). To prove (iv), we use summation by parts to obtain that

$$\sum_{k=2}^{n} r_k p_k(z) = \sum_{k=2}^{n-1} P_k(z)(r_k - r_{k-1}) + P_n(z)\, r_n\,, \tag{8}$$

where $P_n(z) = p_2(z) + \cdots + p_n(z)$ for all $n = 2, 3,\ldots$. From (4) and [4], we have

$$\begin{aligned} |p_2(z) + p_3(z) + \cdots + p_n(z)| &\leqslant d(2) + \cdots + d(n) \\ &\leqslant Cn \ln n \end{aligned} \tag{9}$$

for all n and $|z| \leqslant 1$, where C is some absolute constant. Combining (8) and (9) gives

$$\begin{aligned} \left| \sum_{k=2}^{n} r_k p_k(z) \right| &\leqslant C \left[\sum_{k=2}^{n-1} (r_k - r_{k+1})\, k \ln k + r_n n \ln n \right] \\ &= C \sum_{k=2}^{n-1} r_k [k \ln k - (k-1) \ln(k-1)] \\ &\leqslant C' \sum_{k=2}^{n-1} r_k \ln k, \end{aligned}$$

which implies the convergence of $\sum_{k=0}^{\infty} r_k p_k(z)$ on $\bar{U}$. This completes the proof of Theorem 2.

3. Mean Boundary Value Problems

In this section, we consider the mean boundary value problems and the mean initial value problems for some elementary differential equations Since every real-valued function $f(1, \theta) = f(e^{i\theta})$ on T can be decomposed into a sum of an even function and an odd function:

$$f(e^{i\theta}) = \frac{f(e^{i\theta}) + f(e^{-i\theta})}{2} + \frac{f(e^{i\theta}) - f(e^{-i\theta})}{2},$$

we consider the even functions and odd functions separately.

THEOREM 3. (i) *Let $f(1, \theta)$ be an even function in $C^{1+\epsilon}(T)$ for some $\epsilon > 0$. Then $f = 0$ if the means $s_n(f) = 0$ for $n = 1, 2, \ldots$.*

(ii) *Let $f(1, \theta)$ be ann odd function defined on T. Then $s_n(f) = 0$ for all n.*

To prove (i), we let

$$f(1, \theta) = \sum_{n=-\infty}^{\infty} a_n e^{i2\pi n\theta}.$$

Since f is even, we have $a_n = a_{-n}$ and hence

$$0 = s_n(f) = 2 \sum_{k=1}^{\infty} a_{kn} + a_0 .$$

As $f \in C^{1+\epsilon}(T)$, we can conclude that $a_n = O(1/n^{1+\epsilon})$ and

$$0 = \lim_{n\to\infty} s_n(f) = a_0 .$$

Then it follows from the proof of Theorem A in [1] that $f = 0$. To prove (ii), we observe that if f is odd, then

$$\begin{aligned} s_n(f) &= \frac{1}{n} \sum_{k=1}^{n} f\left(1, \frac{2\pi k}{n}\right) = \frac{1}{n} \sum_{k=1}^{n} f\left(1, \frac{2\pi(n-k)}{n}\right) \\ &= \frac{1}{n} \sum_{k=1}^{n} f\left(1, \frac{-2\pi k}{n}\right) \\ &= -\frac{1}{n} \sum_{k=1}^{n} f\left(1, \frac{2\pi k}{n}\right) = -s_n , \end{aligned}$$

which implies that $s_n(f) = 0$ for all n.

THEOREM 4. *Let $\{\alpha_n\}$ be a sequence of real numbers tending to α with the rate $\alpha_n - \alpha = O(1/n^{2+\epsilon})$ for some $\epsilon > 0$. Then there exists a unique even function $u(r, \theta) = u(r, -\theta)$ in $C^{1+\epsilon'}(\bar{U})$ for some $\epsilon' > 0$ such that*

$$\Delta u = 0 \text{ in } U$$

and

$$\frac{1}{n} \sum_{k=1}^{n} u\left(1, \frac{2\pi k}{n}\right) = \alpha_n$$

for all $n = 1, 2, \ldots$.

The uniqueness of the function $u(r, \theta)$ follows from Theorem 3. It is obvious that the following series

$$\sum_{n=1}^{\infty} (\alpha_n - \alpha)\, u_n(r, \theta) + \alpha,$$

with $u_n(r, \theta) = \sum_{k|n} \mu(n/k)\, r^k \cos k\theta$, converges uniformly on $\bar{U}$ to a function $u(r, \theta)$, whose Fourier coefficients $a_n[u(1, \theta)]$ can be estimated as follows:

$$\begin{aligned} a_n[u(1, \theta)] &= \tfrac{1}{2} \sum_{k=1}^{\infty} (\alpha_k - \alpha) \left[\sum_{m|k} \mu(k/m)\, \delta_{n,m}\right] + \alpha\delta_{n,0} \\ &= \tfrac{1}{2} \sum_{k=1}^{\infty} (\alpha_{kn} - \alpha) + \alpha\delta_{n,0} \\ &= O(1/n^{2+\epsilon}). \end{aligned}$$

Thus, we can conclude that $u(r, \theta) \in C^{1+\epsilon'}(\bar{U})$ for some $\epsilon' > 0$. It is obvious that $\Delta u = 0$ in U and

$$\begin{aligned} s_n[u(1, \theta)] &= \sum_{k=1}^{\infty} (\alpha_k - \alpha)\, s_n[u_k(1, \theta)] + \alpha \\ &= \sum_{k=1}^{\infty} (\alpha_k - \alpha)\, s_n[\mathrm{Re}\, p_k(e^{i\theta})] + \alpha \\ &= \alpha_n . \end{aligned}$$

The proof of Theorem 4 is then completed.

Since the means $s_n(f)$ of an odd function f in $C(T)$ are always zero, we cannot expect to recapture the function f from the means $s_n(f)$. Hence, we also consider the means of the tangential derivatives of f on T.

THEOREM 5. (i) *Let $f(1, \theta)$ be an odd function in $C^{2+\epsilon}(T)$ for some $\epsilon > 0$. Then $f = 0$ if the means of the tangential derivatives $s_n(\partial f/\partial\theta)$ vanish for all n.*

(ii) *Let $f(1, \theta)$ be an even function in $C^1(T)$ then $s_n(\partial f/\partial\theta) = 0$ for all n.*

This theorem follows from Theorem 3 by noting that $\partial f/\partial\theta$ is an even function on T.

THEOREM 6. *Let $\{\beta_n\}$ be a sequence of real numbers satisfying $\beta_n = O(1/n^{2+\epsilon})$ for some $\epsilon > 0$. Then there exists a unique odd function $v(r, \theta) = -v(r, -\theta)$ in $C^{2+\epsilon'}(\overline{U})$ for some $\epsilon' > 0$ such that*

$$\Delta v = 0 \text{ in } U$$

$$\frac{1}{n}\sum_{k=1}^{n} v_\theta\left(1, \frac{2\pi k}{n}\right) = \beta_n$$

for all $n = 1, 2,\ldots$.

The uniqueness of the function $v(r, \theta)$ follows from Theorem 5. It is obvious that

$$\sum_{n=1}^{\infty} \beta_n v_n(r, \theta) = \sum_{n=1}^{\infty} \beta_n \left[\sum_{k|n} \mu\left(\frac{n}{k}\right) \frac{r^k \sin k\theta}{k}\right]$$

converges uniformly to a function v in $\overline{U}$, whose Fourier coefficients $a_n[v(1, \theta)]$ can be similarly estimated as above:

$$\begin{aligned} a_n[v(1, \theta)] &= \tfrac{1}{2}\sum_{k=1}^{\infty} \beta_{kn} \cdot 1/n \\ &= O(1/n^{3+\epsilon}), \end{aligned}$$

which implies that $v(r, \theta) \in C^{2+\epsilon'}(\overline{U})$ for some $\epsilon' > 0$. It can be easily shown that $v(r, \theta)$ satisfies the required conditions in the theorem.

Combining Theorem 4 and Theorem 6, we have the following theorem.

THEOREM 7. *Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences of real numbers converging to α and 0, respectively, with the rates $\alpha_n - \alpha = O(1/n^{3+\epsilon})$ and $\beta_n = O(1/n^{2+\epsilon})$ for some $\epsilon > 0$. Then there exists a unique function $w(r, \theta)$ in $C^{2+\epsilon'}(\overline{U})$ for some $\epsilon' > 0$ such that*

$$\Delta w = 0 \text{ in } U$$

$$\frac{1}{n}\sum_{k=1}^{n} w\left(1, \frac{2\pi k}{n}\right) = \alpha_n$$

$$\frac{1}{n}\sum_{k=1}^{n} w_\theta\left(1, \frac{2\pi k}{n}\right) = \beta_n$$

for all $n = 1, 2, \ldots$. *Furthermore, the series*

$$\sum_{n=1}^{\infty} (\alpha_n - \alpha)\, u_n(r, \theta) + \sum_{n=1}^{\infty} \beta_n v_n(r, \theta) + \alpha$$

converges uniformly to $w(r, \theta)$ *on* $\overline{U}$. *Here,* u_n *and* v_n *are trigonometric polynomials defined as above.*

THEOREM 8. *Let* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *be two sequences of real numbers tending to zero with the rates* $\alpha_n = O(1/n^{3+\epsilon})$ *and* $\beta_n = O(1/n^{2+\epsilon})$ *for some* $\epsilon > 0$. *Then there exists a unique function* $u(r, \theta) \in C^{3+\epsilon'}(\overline{U})$ *for some* $\epsilon' > 0$ *such that*

$$\Delta u = 0 \text{ in } U,$$

$$\frac{1}{n} \sum_{k=1}^{n} u_r \left(1, \frac{2\pi k}{n}\right) = \alpha_n$$

$$\frac{1}{n} \sum_{k=1}^{n} u_{r\theta} \left(1, \frac{2\pi k}{h}\right) = \beta_n$$

for all $n = 1, 2, \ldots,$ *and* $u(0, \theta) = 0$.

From the rates of convergence of the sequences $\{\alpha_n\}$ and $\{\beta_n\}$, we can conclude that the following series

$$\sum_{n=1}^{\infty} \alpha_n \sum_{k|n} \mu\left(\frac{n}{k}\right) \frac{r^k}{k} \cos k\theta + \sum_{n=1}^{\infty} \beta_n \sum_{k|n} \mu\left(\frac{n}{k}\right) \frac{r^k}{k^2} \sin k\theta$$

converges uniformly to a function $u(r, \theta) \in C^{3+\epsilon'}(\overline{U})$ satisfying the required conditions. To prove the uniqueness of u, we assume $\alpha_n = \beta_n = 0$ for all $n = 1, 2, \ldots$. Noting that ru_r is a solution of the following problem:

$$\Delta w = 0 \text{ in } U$$

and

$$\frac{1}{n} \sum_{k=1}^{n} w\left(1, \frac{2\pi k}{n}\right) = \frac{1}{n} \sum_{k=1}^{n} w_\theta \left(1, \frac{2\pi k}{n}\right) = 0,$$

we can conclude from Theorem 7 and the condition $u(0, \theta) = 0$ that $u = 0$. This completes the proof of the theorem.

THEOREM 9. *Let* $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, *and* $\{\delta_n\}$ *be sequences of real numbers converging to* α, 0, 0, *and* 0, *respectively, with the rates* $\alpha_n - \alpha = O(1/n^{3+\epsilon})$,

$\beta_n = O(1/n^{2+\epsilon})$, $\gamma_n = O(1/n^{2+\epsilon})$, and $\delta_n = O(1/n^{1+\epsilon})$, where $\epsilon > 0$. Then there exists a unique function $u(x, t)$ in $C^{2+\epsilon'}(\mathbb{R}^2)$ for some $\epsilon' > 0$ such that

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2},$$

$$1/n \sum_{k=1}^{n} u(j + k/n, 0) = \alpha_n,$$

$$1/n \sum_{k=1}^{n} u_x(j + k/n, 0) = \beta_n,$$

$$1/n \sum_{k=1}^{n} u_t(j + k/n, 0) = \gamma_n,$$

$$1/n \sum_{k=1}^{n} u_{tx}(j + k/n, 0) = \delta_n,$$

for all $j = 0, \pm 1, \ldots$ and $n = 1, 2, \ldots$.

It is clear from the assumptions on $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, and $\{\delta_n\}$ that the following series

$$\sum_{n=1}^{\infty} (\alpha_n - \alpha) \sum_{k|n} \mu\left(\frac{n}{k}\right) \cos 2k\pi x \cos 2k\pi t + \alpha$$
$$+ \sum_{n=1}^{\infty} \beta_n \sum_{k|n} \mu\left(\frac{n}{k}\right) \frac{\sin 2k\pi x}{2k\pi} \cos 2k\pi t$$
$$+ \sum_{n=1}^{\infty} \gamma_n \sum_{k|n} \mu\left(\frac{n}{k}\right) \frac{\cos 2k\pi x}{2k\pi} \sin 2k\pi t$$
$$+ \sum_{n=1}^{\infty} \delta_n \sum_{k|n} \mu\left(\frac{n}{k}\right) \frac{\sin 2k\pi x}{4k^2\pi^2} \sin 2k\pi t$$

converges uniformly to a solution of the mean initial value problem with the required smoothness condition. To prove the uniqueness, we let u and v be two solutions of the mean initial value problem and define f on the unit circle T by

$$f(e^{i2\pi x}) = u(x, 0) - v(x, 0)$$

for $0 \leqslant x \leqslant 1$. Since $u(0, 0) = u(1,0) = v(0, 0) = v(1, 0)$ and $u_x(0, 0) = u_x(1, 0) = v_x(0, 0) = v_x(1, 0)$, we can conclude that the Fourier coefficients $a_n(f)$ of f satisfy $a_n(f) = O(1/n^{2+\epsilon'})$. Thus, $f = 0$ by a proof similar to the proof of Theorem A. Similarly, we can conclude that $u(x, 0) = v(x, 0)$ for all x and, hence, $u(x, t) = v(x, t)$ for all x and t.

Also, we can obtain the following theorem for heat equations.

THEOREM 10. *Let* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *be two sequences of real numbers converging to* α *and* 0, *respectively, with the rates* $\alpha_n - \alpha = O(1/n^{3+\epsilon})$ *and* $\beta_n = O(1/n^{2+\epsilon})$ *for some* $\epsilon > 0$. *Then there exists a unique solution for the following initial value problem*:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \qquad \textit{for all} \quad (x, t) \in (-\infty, \infty) \times (0, \infty),$$

$$\frac{1}{n} \sum_{k=1}^{n} u\left(\frac{k}{n}, 0\right) = \alpha_n \qquad \textit{for all} \quad n = 1, 2, \ldots,$$

$$\frac{1}{n} \sum_{k=1}^{n} u_x\left(\frac{k}{n}, 0\right) = \beta_n \qquad \textit{for all} \quad n = 1, 2, \ldots,$$

$$u(x, 0) = u(x + 1, 0) \in C^{2+\epsilon}(\mathbb{R}).$$

Furthermore, the solution can be represented by the following series:

$$u(x, t) = \alpha + \sum_{n=1}^{\infty} (\alpha_n - \alpha) \sum_{k|n} \mu\left(\frac{n}{k}\right) e^{-4\pi^2 k^2 t} \cos 2\pi k x + \sum_{n=1}^{\infty} \beta_n \sum_{k|n} \mu\left(\frac{n}{k}\right) \frac{\sin 2\pi k x}{2\pi k} e^{-4\pi^2 k^2 t}.$$

4. FINAL REMARKS AND EXTENSIONS

We first remark that the results in the first section can be generalized to a polydisc. To do this we need the following lemma, which can be proved by induction and an application of Theorem A.

LEMMA. *Let* $f(z_1, \ldots, z_N) = \sum a_{m_1 \cdots m_N} z_1^{m_1} \cdots z_N^{m_N}$ *be such that*

$$a_{m_1 \cdots m_N} = O\left(\frac{1}{(m_1 \cdots m_N)^{1+\epsilon}}\right)$$

for some $\epsilon > 0$. *If the arithmetic means* $s_n(f)$ *of* f *defined by*

$$s_n(f) = \frac{1}{n_1 \cdots n_N} \sum_{k_1=1}^{n_1} \cdots \sum_{k_N=1}^{n_N} f(e^{i2\pi k_1/n_1}, \ldots, e^{i2\pi k_N/n_N})$$

are all zero, where $n = (n_1, \ldots, n_N)$, $n_1, \ldots, n_N \geqslant 1$, *then* f *is identically equal to zero.*

For a sequence of complex numbers $c_n = c_{n_1 \cdots n_N}$, if the limit of c_n, as $n_{j_1}, \ldots, n_{j_p}$ tend to infinity exists, we denote the limit by $c_{m_1 \cdots m_N}$ where $m_k = \infty$ if $k = j_q$, $1 \leqslant q \leqslant p$, and $m_k = n_k$ otherwise. We define a sequence of polynomials $p_n(z) = p_{n_1 \cdots n_N}(z_1, \ldots, z_N)$ where $n_1, \ldots, n_N \geqslant 0$ as follows:

$$p_{n_1 \cdots n_N}(z_1, \ldots, z_N) = 1 \qquad \text{if} \quad n_1 = \cdots = n_N = 0$$

$$p_{n_1 \cdots n_{j-1} 0 n_{j+1} \cdots n_N}(z_1, \ldots, z_N) = p_{n_1 \cdots n_{j-1} n_{j+1} \cdots n_N}(z_1, \ldots, z_{j-1}, z_{j+1}, \ldots, z_N)$$

$$p_{m_1 \cdots m_k}(z_1, \ldots, z_k) = \sum_{s_1 | m_1} \cdots \sum_{s_k | m_k} \mu\left(\frac{m_1}{s_1}\right) \cdots \mu\left(\frac{m_k}{s_k}\right) z_1^{s_1} \cdots z_k^{s_k}$$
$$\text{if} \quad m_1, \ldots, m_k \geqslant 1, \quad 1 \leqslant k \leqslant N.$$

For a continuous function f on T^N, the distinguished boundary of the unit polydisc U^N in the space of N complex variables, we define a sequence $r_n(f) = r_{n_1 \cdots n_N}(f)$, which we call the *Riemann coefficients* of f, by

$$r_{m_1 \cdots m_N}(f) = s_{\infty \cdots \infty}(f) \qquad \text{if} \quad m_1 = \cdots = m_N = 0;$$

$$r_{0 \cdots 0 m_j 0 \cdots 0}(f) = s_{\infty \cdots \infty m_j \infty \cdots \infty}(f) - s_{\infty \cdots \infty}(f) \qquad \text{if} \quad m_j \geqslant 1;$$

$$r_{0 \cdots 0 m_j 0 \cdots 0 m_k 0 \cdots 0}(f) = s_{\infty \cdots \infty m_j \infty \cdots \infty m_k \infty \cdots \infty}(f) - s_{\infty \cdots \infty m_j \infty \cdots \infty}(f)$$
$$- s_{\infty \cdots \infty m_k \infty \cdots \infty}(f) + s_{\infty \cdots \infty}(f) \qquad \text{if} \quad m_j, m_k \geqslant 1;$$

$$\cdots$$

$$r_{m_1 \cdots m_N}(f) = s_{m_1' \cdots m_1'} - \sum_1 + \sum_2 - \cdots,$$

where $m_j' = m_j$ if $m_j \neq 0$ and $m_j' = \infty$ if $m_j = 0$ and where $\sum_1$ is the sum of the $s_{m_1' \cdots m_N'}$ with one noninfinity m_j' replaced by infinity, $1 \leqslant j \leqslant N$; and more generally, $\sum_k$ is the sum of the $s_{m_1' \cdots m_N'}$ with k noninfinity m_j' replaced by infinity. By the above lemma and an extension of the proof of Theorem 1, we obtain the following theorem.

THEOREM 11. *Let $f \in C^{1+\epsilon}(T^N)$ for some $\epsilon > 0$ and $r_n(f)$ be the* n*th Riemann coefficients of f. Then the series $\sum_n r_n(f)\, p_n(z)$, where $n = (n_1, \ldots, n_N)$ with $n_j \geqslant 0$ for all j, converges uniformly on the closed unit polydisc $\overline{U}^N$ to a function F. Furthermore, if f admits an analytic continuation in U^N, then $F = f$ on T^N.*

We also remark that in Theorem A we cannot in general move the roots of unity. For instance, if we take $z_{n,k} = e^{i2\pi k/n}$, $k = 1, \ldots, n$, for $n \neq 2$ and $z_{2,1} = z_{2,2} = 1$, then the function $f(z) = z^2 - z$ satisfies $s_n(f) = 0$ for all n. As for a Jordan curve in the complex plane different from a circle, we can take the means of a function at the conformal images of the roots of unity and results analogous to the case of the unit circle also follow. Finally, if the Fourier series of the boundary values are known to be a lacunary series (cf. [5]), we can relax some conditions in all the above theorems. For example, in Theorem 1, we need only assume $\sum_{n=1}^{\infty} |a_n(f)| < \infty$ instead of $a_n(f) = 1/n^{1+\epsilon}$ and obtain the theorem by using a result in [1] and an argument similar to the proof of Theorem 2(iii).

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